



Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

On Certain Partial Differential Equations connected with the Theory of Surfaces.*

BY THOMAS CRAIG.

Let u and v denote the parameters of the lines of curvature of a surface, ρ_1 and ρ_2 the principal radii of curvature of the surface at the point (u, v) , ρ_1 being the principal radius corresponding to the line $v = \text{const.}$ (the u -line), and ρ_2 corresponding to $u = \text{const.}$ (the v -line). Denote further by R_1, R_2 the radii of geodesic curvature of the lines $u = \text{const.}$ and $v = \text{const.}$ respectively; let α, β, γ be the direction cosines of the normal to the surface and write

$$L = \sum \alpha \frac{\partial^2 x}{\partial u^2}, \quad M = \sum \alpha \frac{\partial^2 x}{\partial u \partial v} = 0, \quad N = \sum \alpha \frac{\partial^2 x}{\partial v^2}. \quad (1)$$

We have now

$$\left. \begin{aligned} \rho_1 &= \frac{E}{L}, \quad \rho_2 = \frac{G}{N}, \\ \frac{1}{R_2} &= \frac{-1}{2E\sqrt{G}} \frac{\partial E}{\partial v}, \quad \frac{1}{R_1} = \frac{-1}{2G\sqrt{E}} \frac{\partial G}{\partial u}. \end{aligned} \right\} \quad (2)$$

We have further the following three equations connecting all of the preceding quantities

$$\left. \begin{aligned} \frac{1}{\rho_1^2} \frac{\partial \rho_1}{\partial v} + \frac{\sqrt{G}}{R_2} \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) &= 0, \\ \frac{1}{\rho_2^2} \frac{\partial \rho_2}{\partial u} - \frac{\sqrt{E}}{R_1} \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) &= 0, \end{aligned} \right\} \quad (3)$$

$$\frac{\sqrt{EG}}{\rho_1 \rho_2} = \frac{\partial}{\partial u} \frac{\sqrt{G}}{R_1} + \frac{\partial}{\partial v} \frac{\sqrt{E}}{R_2}. \quad (4)$$

Equations (3) can also be written

$$\left. \begin{aligned} \frac{\partial}{\partial v} \frac{\sqrt{E}}{\rho_1} &= \frac{\sqrt{E}}{R_2} \cdot \frac{\sqrt{G}}{\rho_2}, \\ \frac{\partial}{\partial u} \frac{\sqrt{G}}{\rho_2} &= -\frac{\sqrt{G}}{R_1} \cdot \frac{\sqrt{E}}{\rho_1}. \end{aligned} \right\} \quad (5)$$

* "Sur une suite d'équations linéaires aux dérivées partielles provenant de la théorie des surfaces." *Comptes Rendus*, Oct. 26, 1896, p. 634. In this note the letter ε has been printed throughout for E .

Differentiating the first of (3) for u and the second for v and eliminating $\frac{1}{\rho_2}$ and $\frac{1}{\rho_1}$ respectively, we have

$$\frac{\partial^2}{\partial u \partial v} \frac{1}{\rho_1} - \frac{\sqrt{G}}{R_2} \frac{\partial}{\partial u} \frac{1}{\rho_1} - \left(\frac{\partial}{\partial u} \log \frac{\sqrt{G}}{R_2} + \frac{\sqrt{E}}{R_1} \right) \frac{\partial}{\partial v} \frac{1}{\rho_1} = 0, \quad (6)$$

$$\frac{\partial^2}{\partial u \partial v} \frac{1}{\rho_2} - \left(\frac{\partial}{\partial v} \log \frac{\sqrt{E}}{R_1} + \frac{\sqrt{G}}{R_2} \right) \frac{\partial}{\partial u} \frac{1}{\rho_2} - \frac{\sqrt{E}}{R_1} \frac{\partial}{\partial v} \frac{1}{\rho_2} = 0; \quad (7)$$

that is, $\frac{1}{\rho_1}$ and $\frac{1}{\rho_2}$ are particular integrals of the differential equations

$$(E_{01}) = \frac{\partial^2 \phi_{01}}{\partial u \partial v} - \frac{\sqrt{G}}{R_2} \frac{\partial \phi_{01}}{\partial u} - \left(\frac{\partial}{\partial u} \log \frac{\sqrt{G}}{R_2} + \frac{\sqrt{E}}{R_1} \right) \frac{\partial \phi_{01}}{\partial v} = 0, \quad (8)$$

$$(E_{02}) = \frac{\partial^2 \phi_{02}}{\partial u \partial v} - \left(\frac{\partial}{\partial v} \log \frac{\sqrt{E}}{R_1} + \frac{\sqrt{G}}{R_2} \right) \frac{\partial \phi_{02}}{\partial u} - \frac{\sqrt{E}}{R_1} \frac{\partial \phi_{02}}{\partial v} = 0. \quad (9)$$

The differential equation satisfied by the coordinates (x, y, z) of the point (u, v) of the surface is

$$\frac{\partial^2 \phi_0}{\partial u \partial v} + \frac{\sqrt{G}}{R_2} \frac{\partial \phi_0}{\partial u} + \frac{\sqrt{E}}{R_1} \frac{\partial \phi_0}{\partial v} = 0. \quad (10)$$

From equations (5) we derive in the same way as above the following:

$$\frac{\partial^2 \phi_{01}}{\partial u \partial v} - \frac{\partial}{\partial u} \log \frac{\sqrt{E}}{R_2} \cdot \frac{\partial \phi_{01}}{\partial v} - \frac{\sqrt{EG}}{R_1 R_2} \phi_{01} = 0, \quad (11)$$

$$\frac{\partial^2 \phi_{02}}{\partial u \partial v} - \frac{\partial}{\partial v} \log \frac{\sqrt{G}}{R_1} \cdot \frac{\partial \phi_{02}}{\partial u} - \frac{\sqrt{EG}}{R_1 R_2} \phi_{02} = 0. \quad (12)$$

These are *equivalent* to (8) and (9) (that is, have the same invariants, or are derived from (8) and (9) by substitutions of the form $\phi = \lambda \phi'$) and have

$\frac{\sqrt{E}}{\rho_1}$ and $\frac{\sqrt{G}}{\rho_2}$ respectively as particular integrals.

The invariants of (E_{01}) are

$$h_{01} = \frac{\sqrt{EG}}{R_1 R_2}, \quad k_{01} = -\frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{E}}{R_2} + \frac{\sqrt{EG}}{R_1 R_2}; \quad (13)$$

those of (E_{02}) are

$$h_{02} = -\frac{\partial^2}{\partial u \partial v} \log \frac{\sqrt{G}}{R_1} + \frac{\sqrt{EG}}{R_1 R_2}, \quad k_{02} = \frac{\sqrt{EG}}{R_1 R_2}. \quad (14)$$

Form now the Laplace's series,

$$\left. \begin{aligned} & \dots (E_{-11}), \dots (E_{-21}), (E_{-11}), (E_{01}), (E_{11}), (E_{21}) \dots, (E_{11}) \dots \\ & \dots (E_{-12}), \dots (E_{-22}), (E_{-12}), (E_{02}), (E_{12}), (E_{22}) \dots, (E_{12}) \dots \end{aligned} \right\} \quad (15)$$

It is easy to see by calculation of the invariants that these two series of equations are equivalent, viz. the invariants of $(E_{k-1, 2})$ and $(E_{k, 1})$ are the same, or, what comes to the same thing, the equations in the lower line are transformed into those in the upper line by substitutions of the form $z = \lambda z'$, denoting by z the unknown function and by λ a function of (u, v) . In $(E_{k-1, 2})$ replace $\phi_{k-1, 2}$ by $\frac{\sqrt{G}}{R_2} \phi'_{k-1, 2}$, the new equation in $\phi'_{k-1, 2}$ will be $(E_{k, 1})$, that is, $\phi'_{k-1, 2} \equiv \phi_{k, 1}$. This is for k positive. For the negative values of k we write for $\phi_{-k, 2}$ the quantity $\frac{\sqrt{E}}{R_1} \phi'_{-k, 2}$ and so transform $(E_{-k, 2})$ into $(E_{-(k-1), 1})$.

We find readily the following relations for the invariants:

$$\left. \begin{aligned} h_{i1} = h_{i-1, 2} &= \frac{\sqrt{EG}}{R_1 R_2} - \frac{\partial^2}{\partial u \partial v} \log \left(\frac{G^{\frac{i}{2}}}{R_1^i} h_{11}^{i-1} h_{21}^{i-2} \dots h_{i-1, 1} \right), \\ k_{i1} = k_{i-1, 2} &= \frac{\sqrt{EG}}{R_1 R_2} - \frac{\partial^2}{\partial u \partial v} \log \left(\frac{G^{\frac{i-1}{2}}}{R_1^{i-1}} h_{11}^{i-2} h_{21}^{i-3} \dots h_{i-2, 1} \right), \end{aligned} \right\} \quad (16)$$

$$\left. \begin{aligned} h_{-i2} = h_{-(i-1), 1} &= \frac{\sqrt{EG}}{R_1 R_2} - \frac{\partial^2}{\partial u \partial v} \log \left(\frac{E^{\frac{i-1}{2}}}{R_2^{i-1}} k_{-12}^{i-2} k_{-22}^{i-3} \dots k_{-(i-2), 2} \right), \\ k_{-i2} = k_{-(i-1), 1} &= \frac{\sqrt{EG}}{R_1 R_2} - \frac{\partial^2}{\partial u \partial v} \log \left(\frac{E^{\frac{i}{2}}}{R_2^i} k_{-12}^{i-1} k_{-22}^{i-2} \dots k_{-(i-1), 2} \right), \end{aligned} \right\} \quad (17)$$

Dropping the letter k and using only h (i. e. $h_{-1} = k$), these are

$$h_{i-1, 2} = h_{i1} = \frac{\sqrt{EG}}{R_1 R_2} - \frac{\partial^2}{\partial u \partial v} \log \left(\frac{G^{\frac{i}{2}}}{R_1^i} h_{11}^{i-1} h_{21}^{i-2} \dots h_{i-1, 1} \right), \quad (18)$$

or
$$h_{i-1, 2} = h_{i1} = \frac{\sqrt{EG}}{R_1 R_2} - \frac{\partial^2}{\partial u \partial v} \log \left(\frac{G^{\frac{i}{2}}}{R_1^i} h_{02}^{i-1} h_{12}^{i-2} \dots h_{i-2, 2} \right). \quad (18')$$

Also
$$h_{-(i+1), 2} = h_{-i1} = \frac{\sqrt{EG}}{R_1 R_2} - \frac{\partial^2}{\partial u \partial v} \log \left(\frac{E^{\frac{i}{2}}}{R_2^i} h_{-11}^{i-1} h_{-21}^{i-2} \dots h_{-(i-1), 2} \right), \quad (19)$$

or
$$h_{-(i+1), 2} = h_{-i1} = \frac{\sqrt{EG}}{R_1 R_2} - \frac{\partial^2}{\partial u \partial v} \log \left(\frac{E^{\frac{i}{2}}}{R_2^i} h_{-22}^{i-1} h_{-32}^{i-2} \dots h_{-i2} \right). \quad (19')$$

Before considering the general case, we shall examine the case where equations (8) and (9) become identical. For this we must have

$$\frac{\partial}{\partial u} \log \frac{\sqrt{G}}{R_2} = 0, \quad \frac{\partial}{\partial u} \log \frac{\sqrt{E}}{R_1} = 0. \quad (20)$$

From these we find readily

$$E = U_1 V_2, \quad G = U_2 V_1, \quad (21)$$

the U 's being functions of u alone and the V 's functions of v alone. We have now for the linear element

$$ds^2 = U_2 V_2 \left(\frac{U_1}{U_2} du^2 + \frac{V_1}{V_2} dv^2 \right), \quad (22)$$

or say

$$ds^2 = \frac{1}{\lambda} (U du^2 + V dv^2). \quad (23)$$

In this case then the lines of curvature form an isometric system. The differential equation satisfied by $\frac{1}{\rho_1}$ and $\frac{1}{\rho_2}$ is now readily seen to take the form

$$\frac{\partial^2 \phi}{\partial u \partial v} - \frac{\partial \log \lambda^{\frac{1}{2}}}{\partial v} \frac{\partial \phi}{\partial u} - \frac{\partial \log \lambda^{\frac{1}{2}}}{\partial u} \frac{\partial \phi}{\partial v} = 0, \quad (24)$$

and that satisfied by the coordinates x, y, z of a point on the surface is

$$\frac{\partial^2 \theta}{\partial u \partial v} + \frac{\partial \log \lambda^{\frac{1}{2}}}{\partial v} \frac{\partial \theta}{\partial u} + \frac{\partial \log \lambda^{\frac{1}{2}}}{\partial u} \frac{\partial \theta}{\partial v} = 0. \quad (25)$$

This is the adjoint equation to (24), since λ is of the form $\frac{1}{U_2(u) V_2(v)}$, or say simply $U_2 V_2$, for now $\frac{\partial^2 \log \lambda}{\partial u \partial v} = 0$.

In the case then *when the lines of curvature of a surface form an isometric system, and the linear element has the form*

$$G^2 = U_2(u) V_2(v) (U du^2 + V dv^2),$$

the partial differential equation satisfied by the reciprocals of the principal radii of curvature is adjoint to that satisfied by the cartesian coordinates of a point of the surface.

The common value of the invariants of both equations is

$$h = k = \frac{\partial \log \lambda^{\frac{1}{2}}}{\partial u} \frac{\partial \log \lambda^{\frac{1}{2}}}{\partial v} = \frac{1}{4} \frac{U_2'(u)}{U_2(u)} \cdot \frac{V_2'(v)}{V_2(v)}. \quad (26)$$

It is not my purpose here to examine the general system of equations implied in (15), but we may look for a moment at the first of the Laplace's equations derived from (24); this is

$$\frac{\partial^2 \phi_1}{\partial u \partial v} - \frac{\partial \log \lambda^{\frac{1}{2}} h}{\partial v} \frac{\partial \phi_1}{\partial u} - \frac{\partial \log \lambda^{\frac{1}{2}}}{\partial u} \frac{\partial \phi_1}{\partial v} + \frac{\partial \log \lambda^{\frac{1}{2}}}{\partial u} \frac{\partial \log h}{\partial v} \phi_1 = 0. \quad (27)$$

The invariants are

$$\left. \begin{aligned} h_1 &= \frac{\partial \log \lambda^\dagger}{\partial u} \frac{\partial \log \lambda^\dagger}{\partial v} - \frac{\partial^2 \log h}{\partial u \partial v} \\ &= h - \frac{\partial^2 \log h}{\partial u \partial v}, \\ k_1 &= h. \end{aligned} \right\} \quad (28)$$

But

$$h = \frac{1}{4} \frac{\partial \log \lambda}{\partial u} \frac{\partial \log \lambda}{\partial v} = \frac{1}{4} \frac{U'_2}{U_2} \cdot \frac{V'_2}{V_2};$$

therefore

$$\frac{\partial^2 \log h}{\partial u \partial v} = 0,$$

and consequently

$$h_1 = k_1 = h = k.$$

The same thing will clearly be true of all the following equations of the positive end of Laplace's series, and from the known properties of this series in the case when the leading equation has equal invariants we shall have all the invariants of the negative end of the series equal and equal to the single invariant of the leading equation. All the equations of Laplace's series are therefore equivalent in this case, a fact which is indeed almost evident *a priori*. An extended study of equations (8) and (9) and the corresponding Laplace's series (15) will be undertaken in another paper.